

**RESULTS ON UNIQUENESS OF CERTAIN TYPE OF
NON-LINEAR DIFFERENTIAL POLYNOMIALS
SHARING A VALUE**

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Abstract: In the paper we discuss the distribution of uniqueness and its elements over the extended complex plane from different polynomials of view. We obtain a new result regarding the structure and position of uniqueness. This new result has immense application like classifying different expressions to be or not to be unique. The principle objective of the paper is to study the uniqueness of meromorphic functions when sharing a value with weight $l(\geq 0)$ and its nonlinear differential polynomials. We prove a result which significantly generalize the result of Waghamore and Maligi [Commun. Math. **28** (2020), 289-299] by considering the difference polynomials of the form $f^n(f-1)^mP[f]$ and citing two proper examples we have shown that the result is true only for a particular case. Finally we present the compact version of the same result as an improvement.

Keywords and Phrases: Uniqueness, Entire function, difference-differential polynomial.

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1. Introduction

In this paper, we shall always mean meromorphic function in the complex plane. We assume that the reader is familiar with the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [6, 12, 17]. For a non constant meromorphic function f , we denote by $T(r, f)$ the nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$, possibly outside of a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ as $r \rightarrow \infty$, possibly outside of a set of finite linear measure. We say that the meromorphic function $\alpha(z)$ is a small function of f , if $T(r, \alpha(z)) = S(r, f)$.

Let $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM (Counting Multiplicities). If $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM (Ignoring Multiplicities). In addition, we need the following definitions.

Definition 1.1. [8] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f)$ the counting function of simple a -points of f . For a positive integer k we denote by $N_{(k)}(r, a; f)$ the counting function of those a -points of f (counted with proper multiplicities) whose multiplicities are not greater than k . By $\overline{N}_{(k)}(r, a; f)$ we denote the corresponding reduced counting function. Analogously, we can define $N_{(k)}(r, a; f)$ and $\overline{N}_{(k)}(r, a; f)$.

Definition 1.2. [9] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a -points of f , where an a -points of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_{(2)}(r, a; f) + \dots + \overline{N}_{(k)}(r, a; f)$$

clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.3. A differential polynomial $P[f]$ of a nonconstant meromorphic function f is defined as

$$P[f] := \sum_{i=1}^m M_i[f]$$

where $M_i[f] = a_i \prod_{j=0}^k (f^{(j)})^{n_{ij}}$ with $n_{i0}, n_{i1}, \dots, n_{ik}$ as nonnegative integers and $a_i (\neq 0)$ are meromorphic functions satisfying $T(r, a_i) = o(T(r, f))$ as $r \rightarrow \infty$.

The numbers

$$\bar{d}(P) = \max_{1 \leq i \leq m} \sum_{j=0}^k n_{ij}$$

and

$$\underline{d}(P) = \min_{1 \leq i \leq m} \sum_{j=0}^k n_{ij}$$

are respectively called the degree and lower degree of $P[f]$. If $\bar{d}(P) = \underline{d}(P) = d$ (say), then we say that $P[f]$ is a homogeneous differential polynomial of degree d .

Also we define $Q = \max_{1 \leq i \leq m} \{n_{i0} + n_{i1} + 2n_{i2} + \dots + kn_{ik}\}$.

Recently, the exploration of difference polynomials has garnered significant attention within the academic literature. A notable current trend in this field revolves around the investigation of uniqueness results for differential-difference polynomials of meromorphic functions, which encompass both derivatives and differences. A lot of works on entire and meromorphic functions whose differential polynomials generated by them share certain value, small function or fixed points have been done by many mathematicians [1, 2, 4, 5, 10, 18] throughout the world. In [7] the following question was asked: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of nonlinear differential polynomials is remarkable and continuous efforts are being put in to relax the hypothesis of the results (see [1, 4, 5, 11, 13]).

In 2011, Dyavanal [3] proved the following results, which to the knowledge of the authors probably are the first approach in which in order to consider the value sharing of two differential polynomials the multiplicities of zeros and poles of f and g are taken into account.

Theorem A. *Let f and g be two nonconstant meromorphic functions with zeros and poles of multiplicities at least s , where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n + 1)s \geq 12$. If $f^n f'$ and $g^n g'$ share 1 CM, then $f(z) \equiv dg(z)$ for some $(n + 1)$ -th roots of unity d , or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

In 2020, H. P. Waghmare and R. Maligi [14] proved the following results.

Theorem B. *Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s . Suppose that $f^n(f - 1)^m f^{(k)}$ and*

$g^n(g-1)^m g^{(k)}$ share 1 IM, where s , n and k are positive integers. If $(n-m-4)s \geq 4k+18$ for $s=1,2$ and $(n-m)s \geq 8k+23$ for $s \geq 3$, then

1. $f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m+1, n+m, \dots, n+1)$.
2. $f^n(f-1)^m f^{(k)} = g^n(g-1)^m g^{(k)}$, if $\frac{f}{g}$ is not a constant.

Theorem C. Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s , where $s(\geq 7)$ is a positive integer. Let n and k be an integer satisfying $(n-m)s \geq 2k+11$. If $f^n(f-1)^m f^{(k)}$ and $g^n(g-1)^m g^{(k)}$ share 1 CM, then

1. $f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m+1, n+m, \dots, n+1)$.
2. $f^n(f-1)^m f^{(k)} = g^n(g-1)^m g^{(k)}$, if $\frac{f}{g}$ is not a constant.

Regarding Theorems B-C, one may ask the following question which is the motivation of the present paper.

Question 1.1. What can be said about the meromorphic functions f and g if we consider the difference polynomials of the form $f^n(f-1)^m P[f]$, where $P[f]$ is defined in Definition 1.3?

In the paper, our main concern is to find the possible answer of the above question.

For the generalizations and improvements of Theorems B-C to a large extent, we obtain the following results, which are the main results of the present paper.

Theorem 1.1. Let f and g be two nonconstant meromorphic functions with multiplicities of zeros and poles no less than s , where s is a positive integer. Suppose that n , m , $\bar{d}(P)$, $\underline{d}(P)$, k and Q are positive integers. Let $f^n(f-1)^m P[f]$ and $g^n(g-1)^m P[g]$ share $(1, l)$, where $l(\geq 0)$ is an integer. Then we have the following cases.

(i) $l = 0$, $n > m + 8\bar{d}(P) - 4\underline{d}(P) + \frac{4Q+17}{s}$ for $s = 1, 2$. and $n > m + \frac{(4k+5)\bar{d}(P)+4Q+17}{s}$ for $s \geq 3$.

(ii) $l = 1$ and $n > m + \frac{(3k+5)\bar{d}(P)+3Q+19}{2s}$ for $s \geq 7$.

(iii) $l \geq 2$ and $n > m + \frac{(k+2)\bar{d}(P)+Q+8}{s}$ for $s \geq 7$.

Remark 1.1. In Theorem 1.1 giving specific values for s , m , Q and $\bar{d}(P) = \underline{d}(P) = 1$ in the condition $(n-m-8\bar{d}(P)+4\underline{d}(P))s > 4Q+17$, we get the following interesting case:

Case 1. If $s = 1, 2$, $m = m$ and $Q = k$, the result reduces to Theorem B, i.e., $(n-m-4)s \geq 4k+18$.

Similarly for the condition $(n-m)s > \bar{d}(P)(4k+5)+4Q+17$, we get the following interesting case:

Case 2. If $s \geq 3$, $m = m$ and $Q = k$, the result reduces to Theorem B, i.e.,

$$(n - m)s \geq 8k + 23.$$

Again, in Theorem 1.1 giving specific values for s , m , Q and $\bar{d}(P) = 1$ in the condition $(n - m)s > \bar{d}(P)(k + 2) + Q + 8$, we get the following interesting case:

Case 3. If $s \geq 7$, $m = m$ and $Q = k$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem C, i.e., $(n - m)s \geq 2k + 11$.

Example 1.1. Let $f(z) = e^z$, $g(z) = te^z$, where $t^{m+n+1} = 1$, $m, n \in \mathbb{N}$. Let $P[f] = f(z)$ then it is easy to see that the conclusions of theorem 1.1 hold.

Example 1.2. Let $f(z) = \frac{e^z}{e^{2\pi zi/c} - 1}$, $g(z) = tf(z)$, where $t^{m+n+1} = 1$, $m, n \in \mathbb{N}$. Let $P[f] = f(z)$ then it is easy to see that the conclusions of theorem 1.1 hold.

2. Lemmas

In this section, we state some lemmas which will be needed in the sequel. We denote by H the following function :

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1} \right)$$

where F and G are nonconstant meromorphic functions defined in the complex plane \mathbb{C} .

Lemma 2.1. [16] *Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f . Then*

$$T(r, P[f]) \leq \bar{d}(P)T(r, f) + Q\bar{N}(r, \infty; f) + S(r, f).$$

Lemma 2.2. [22] *Let f be a nonconstant meromorphic function and let p, k be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f) \quad (2.1)$$

and

$$N_p(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \quad (2.2)$$

Lemma 2.3. [15] *Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f . Then*

$$m \left(r, \frac{P[f]}{f^{\bar{d}(P)}} \right) \leq (\bar{d}(P) - \underline{d}(P))m(r, 0; f) + S(r, f), \quad (2.3)$$

$$N \left(r, \frac{P[f]}{f^{\bar{d}(P)}} \right) \leq (\bar{d}(P) - \underline{d}(P))N(r, 0; f) + Q[\bar{N}(r, \infty; f) + \bar{N}(r, 0; f)] + S(r, f) \quad (2.4)$$

and

$$N(r, 0; P[f]) \leq (\bar{d}(P) - \underline{d}(P))m(r, 0; f) + Q\bar{N}(r, \infty; f) + N(r, 0; f^{\bar{d}(P)}) + S(r, f). \quad (2.5)$$

Lemma 2.4. [15] *Let f be a nonconstant meromorphic function and $P[f]$ be a differential polynomial of f . Then*

$$N_p(r, 0; P[f]) \leq \bar{d}(P)N_{p+k}(r, 0; f) + Q\bar{N}(r, \infty; f) + S(r, f) \quad (2.6)$$

and

$$N_p(r, 0; P[f]) \leq T(r, P[f]) - \bar{d}(P)T(r, f) + \bar{d}(P)N_{p+k}(r, 0; f) + S(r, f). \quad (2.7)$$

Lemma 2.5. [18, 19] *Let f and g be two nonconstant meromorphic functions sharing (1, 2). Then we have one of the following three cases:*

1. $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g);$
2. $f(z) \equiv g(z);$
3. $f(z)g(z) \equiv 1.$

Lemma 2.6. [20] *Let f and g be two nonconstant meromorphic functions that share (1, m) and $H \neq 0$. Then we have the following two cases:*

1. *If $m = 1$, then*

$$\begin{aligned} T(r, f) \leq & N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) \\ & + \frac{1}{2}\bar{N}(r, \infty; f) + \frac{1}{2}\bar{N}(r, 0; f) + S(r, f) + S(r, g). \end{aligned}$$

2. *If $m = 0$, then*

$$\begin{aligned} T(r, f) \leq & N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) \\ & + 2\bar{N}(r, \infty; f) + \bar{N}(r, \infty; g) + 2\bar{N}(r, 0; f) + \bar{N}(r, 0; g) \\ & + S(r, f) + S(r, g). \end{aligned}$$

Lemma 2.7. [19] *Let f and g be two nonconstant meromorphic functions and $H \equiv 0$. If*

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\bar{N}(r, 0; f) + \bar{N}(r, 0; g) + \bar{N}(r, \infty; f) + \bar{N}(r, \infty; g)}{T(r)} < 1$$

then $f = g$ or $fg = 1$, where E is a set of finite linear measure and not necessarily the same at each of its occurrence.

Lemma 2.8. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s , where s is a positive integer. Let $P[f]$ and Q , be defined as in Definition 1.3, and let $n, m, \underline{d}(P)$ and $\bar{d}(P)$ be positive integers. Then*

$$f^n(f-1)^m P[f] g^n(g-1)^m P[g] \neq 1.$$

Proof. Suppose that

$$f^n(f-1)^m P[f] g^n(g-1)^m P[g] \equiv 1. \quad (2.8)$$

Let $z_0 (\neq 0, \infty)$ be a zero of f of order p . From (2.8) we know that z_0 is a pole of g . Suppose that z_0 is a pole of g of order q . From (2.8) we obtain

$$np + p\bar{d}(P) - Q = nq + mq + q\bar{d}(P) + Q,$$

that is, $(n + \bar{d}(P))(p - q) = mq + 2Q$, which implies that $p \geq q + 1$ and $mq + 2Q \geq n + \bar{d}(P)$.

Hence,

$$p \geq \frac{n + m + \bar{d}(P) - 2Q}{m}. \quad (2.9)$$

Let $z_1 (\neq 0, \infty)$ be a zero of $(f - 1)$ of order p_1 , then from (2.8), z_1 is a pole of g of order q_1 . Again by (2.8), we get

$$mp_1 + p_1\bar{d}(P) - Q = nq_1 + mq_1 + q_1\bar{d}(P) + Q$$

that is,

$$p_1 \geq \frac{(n + m + \bar{d}(P))s + 2Q}{m + \bar{d}(P)}. \quad (2.10)$$

Let $z_2 (\neq 0, \infty)$ be a zero of f' of order p_2 , that is not a zero of $f(f - 1)$, then from (2.8), z_2 is a pole of g of order q_2 . Again by (2.8), we get

$$\bar{d}(P)(p_2 + 1) - Q = nq_2 + mq_2 + q_2\bar{d}(P) + Q$$

that is,

$$p_2 \geq \frac{(n + m + \bar{d}(P))s + 2Q - \bar{d}(P)}{\bar{d}(P)}. \quad (2.11)$$

In the same manner as above, we have the similar results for the zeros of $g^n(g-1)^m P[g]$. From (2.8) we can write

$$\overline{N}(r, \infty; f^n(f-1)^m P[f]) = \overline{N}(r, 0; g^n(g-1)^m P[g]),$$

i.e.,

$$\overline{N}(r, \infty; f) = \overline{N}(r, 0; g^n(g-1)^m P[g]) = \overline{N}(r, 0; g) + \overline{N}(r, 0; g-1) + \overline{N}(r, 0; P[g])$$

From (2.9) to (2.11), we obtain

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \frac{m}{n+m+\overline{d}(P)-2Q} N(r, 0; g) + \frac{m+\overline{d}(P)}{(n+m+\overline{d}(P))s+2Q} N(r, 0; g-1) \\ &\quad + \frac{\overline{d}(P)}{(n+m+\overline{d}(P))s+2Q-\overline{d}(P)} N(r, 0; P[g]), \\ \overline{N}(r, \infty; f) &\leq \left(\frac{m}{n+m+\overline{d}(P)-2Q} + \frac{m+\overline{d}(P)}{(n+m+\overline{d}(P))s+2Q} \right. \\ &\quad \left. + \frac{\overline{d}(P)(2\overline{d}(P)-\underline{d}(P)+Q)}{(n+m+\overline{d}(P))s+2Q-\overline{d}(P)} \right) T(r, g) + S(r, g). \end{aligned} \quad (2.12)$$

By the second fundamental theorem and (2.12), we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, 0; f-1) + S(r, f), \\ T(r, f) &\leq \left(\frac{m}{n+m+\overline{d}(P)-2Q} + \frac{m+\overline{d}(P)}{(n+m+\overline{d}(P))s+2Q} \right. \\ &\quad \left. + \frac{\overline{d}(P)(2\overline{d}(P)-\underline{d}(P)+Q)}{(n+m+\overline{d}(P))s+2Q-\overline{d}(P)} \right) T(r, g) \\ &\quad + \left(\frac{m}{n+m+\overline{d}(P)-2Q} + \frac{m+\overline{d}(P)}{(n+m+\overline{d}(P))s+2Q} \right) T(r, f) \\ &\quad + S(r, f) + S(r, g) \end{aligned} \quad (2.13)$$

and a similar inequality for $T(r, g)$. Combining the two inequalities, we get

$$\begin{aligned} T(r, f) + T(r, g) &\leq \left(\frac{2m}{n+m+\overline{d}(P)-2Q} + \frac{2(m+\overline{d}(P))}{(n+m+\overline{d}(P))s+2Q} \right. \\ &\quad \left. + \frac{\overline{d}(P)(2\overline{d}(P)-\underline{d}(P)+Q)}{(n+m+\overline{d}(P))s+2Q-\overline{d}(P)} \right) [T(r, f) + T(r, g)] \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (2.14)$$

Giving specific values for $m, k, Q, \underline{d}(P)$ and $\bar{d}(P)$ which satisfy

$$n > m + 8\bar{d}(P) - 4\underline{d}(P) + \frac{4Q + 17}{s}$$

for $s = 1, 2$, we obtain

$$T(r, f) + T(r, g) \leq (0.5258)[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Similarly,

$$n > m + \frac{(4k + 5)\bar{d}(P) + 4Q + 17}{s}$$

for $s \geq 3$, we have

$$T(r, f) + T(r, g) \leq (0.5456)[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Thus (2.14) leads to a contradiction.

This proves the lemma.

3. Proof of Theorem

Proof of Theorem 1.1. Let $F = f^n(f - 1)^m P[f]$ and $G = g^n(g - 1)^m P[g]$. Then F and G share $(1, l)$, where $l(\geq 0)$ is an integer. Now we discuss following three cases separately.

Case 1. Suppose that $l = 0$ and $H \neq 0$. Then by Lemma 2.6(2), we consider three cases.

Subcase 1.1. Suppose that

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ &\quad + 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + 2\bar{N}(r, 0; F) + \bar{N}(r, 0; G) \\ &\quad + S(r, F) + S(r, G). \end{aligned} \tag{3.1}$$

We deduce from (3.1) that

$$\begin{aligned} T(r, F) &\leq 6\bar{N}(r, 0; f) + mN(r, 0; f) + N_2(r, 0; P[f]) + 4\bar{N}(r, 0; g) \\ &\quad + mN(r, 0; g) + N_2(r, 0; P[g]) + 4\bar{N}(r, \infty; f) + 3\bar{N}(r, \infty; g) \\ &\quad + 2\bar{N}(r, 0; P[f]) + \bar{N}(r, 0; P[g]) + S(r, f) + S(r, g). \end{aligned} \tag{3.2}$$

Obviously

$$N(r, \infty; F) = (n + m + \bar{d}(P))N(r, \infty; f) + Q\bar{N}(r, \infty; f) + S(r, f). \tag{3.3}$$

Again we have,

$$\begin{aligned}
 (n+m)m(r, \infty; f) &= m\left(r, \frac{F}{P[f]}\right) \leq m(r, \infty; F) + m(r, 0; P[f]) + S(r, f) \\
 &= m(r, \infty; F) + T(r, P[f]) - N(r, 0; P[f]) + S(r, f) \\
 &\leq m(r, \infty; F) + \bar{d}(P)T(r, f) + Q\bar{N}(r, \infty; f) \\
 &\quad - N(r, 0; P[f]) + S(r, f). \tag{3.4}
 \end{aligned}$$

It follows from (3.2), (3.3) and (3.4) that,

$$\begin{aligned}
 (n+m-\bar{d}(P))T(r, f) &\leq T(r, F) - \bar{d}(P)N(r, \infty; f) - N(r, 0; P[f]) + S(r, f) \\
 &\leq 6\bar{N}(r, 0; f) + mN(r, 0; f) + 4\bar{N}(r, 0; g) + mN(r, 0; g) \\
 &\quad + N_2(r, 0; P[g]) + 4\bar{N}(r, \infty; f) + 3\bar{N}(r, \infty; g) + 2\bar{N}(r, 0; P[f]) \\
 &\quad + \bar{N}(r, 0; P[g]) - \bar{d}(P)N(r, \infty; f) + S(r, f) + S(r, g). \tag{3.5}
 \end{aligned}$$

Using Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
 N_2(r, 0; P[g]) + \bar{N}(r, 0; P[g]) &\leq 2Q\bar{N}(r, \infty; g) + 2\bar{d}(P)N(r, 0; g) \\
 &\quad + 2(\bar{d}(P) - \underline{d}(P))T(r, g) + S(r, g) \\
 &\leq 2\left(\frac{Q}{s} + \bar{d}(P) + \bar{d}(P) - \underline{d}(P)\right)T(r, g) + S(r, g) \\
 &\leq 2\left(2\bar{d}(P) - \underline{d}(P) + \frac{Q}{s}\right)T(r, g) + S(r, g) \tag{3.6}
 \end{aligned}$$

and

$$2\bar{N}(r, 0; P[f]) \leq 2\left(2\bar{d}(P) - \underline{d}(P) + \frac{Q}{s}\right)T(r, f) + S(r, f). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we get

$$\begin{aligned}
 (n+m-\bar{d}(P))T(r, f) &\leq \left(\frac{2Q+10}{s} + m + 3\bar{d}(P) - 2\underline{d}(P)\right)T(r, f) \\
 &\quad + \left(\frac{2Q+7}{s} + m + 4\bar{d}(P) - 2\underline{d}(P)\right)T(r, g) + S(r, f) + S(r, g). \tag{3.8}
 \end{aligned}$$

Similarly for G, we get

$$(n+m-\bar{d}(P))T(r, g) \leq \left(\frac{2Q+10}{s} + m + 3\bar{d}(P) - 2\underline{d}(P)\right)T(r, g)$$

$$+ \left(\frac{2Q + 7}{s} + m + 4\bar{d}(P) - 2\underline{d}(P) \right) T(r, f) + S(r, f) + S(r, g). \quad (3.9)$$

Combining (3.8) and (3.9), we obtain

$$\left(n - m - 8\bar{d}(P) + 4\underline{d}(P) - \frac{4Q + 17}{s} \right) [T(r, f) + T(r, g)] \leq S(r, f) + S(r, g)$$

which contradicts the condition

$$n > m + 8\bar{d}(P) - 4\underline{d}(P) + \frac{4Q + 17}{s}$$

for $s = 1, 2$. Again using Lemmas 2.3 and 2.4, we get

$$\begin{aligned} N_2(r, 0; P[g]) + \bar{N}(r, 0; P[g]) &\leq Q\bar{N}(r, \infty; g) + \bar{d}(P)N_{k+2}(r, 0; g) \\ &\quad + Q\bar{N}(r, \infty; g) + \bar{d}(P)N_{k+1}(r, 0; g) + S(r, g) \\ &\leq \left(\frac{\bar{d}(P)(2k + 3) + 2Q}{s} \right) T(r, g) + S(r, g) \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} 2\bar{N}(r, 0; P[f]) &\leq 2Q\bar{N}(r, \infty; f) + 2\bar{d}(P)N_{k+1}(r, 0; f) \\ &\leq \left(\frac{2\bar{d}(P)(k + 1) + 2Q}{s} \right) T(r, f) + S(r, f). \end{aligned} \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.5), we obtain

$$\begin{aligned} (n + m - \bar{d}(P)) T(r, f) &\leq \left(\frac{2\bar{d}(P)(k + 1) + 2Q + 10}{s} + m - \bar{d}(P) \right) T(r, f) \\ &\quad + \left(\frac{\bar{d}(P)(2k + 3) + 2Q + 7}{s} + m \right) T(r, g) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (3.12)$$

Similarly for G, we obtain

$$\begin{aligned} (n + m - \bar{d}(P)) T(r, g) &\leq \left(\frac{2\bar{d}(P)(k + 1) + 2Q + 10}{s} + m - \bar{d}(P) \right) T(r, g) \\ &\quad + \left(\frac{\bar{d}(P)(2k + 3) + 2Q + 7}{s} + m \right) T(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (3.13)$$

Combining (3.12) and (3.13), we obtain

$$\left(n - m - \frac{\bar{d}(P)(4k + 5) + 4Q + 17}{s}\right) [T(r, f) + T(r, g)] \leq S(r, f) + S(r, g)$$

which contradicts the condition

$$n > m + \frac{(4k + 5)\bar{d}(P) + 4Q + 17}{s}$$

for $s \geq 3$.

Therefore $H \equiv 0$. Since f and g are transcendental meromorphic functions, we have

$$\begin{aligned} (n + m)T(r, f) &= T(r, f^n(f - 1)^m) + S(r, f) \\ &\leq T(r, f^n(f - 1)^m P[f]) + T(r, P[f]) + S(r, f) \\ &\leq T(r, F) + (\bar{d}(P) + Q)T(r, f) + S(r, f) \end{aligned}$$

and so

$$T(r, F) \geq (n + m - \bar{d}(P) - Q)T(r, f) + S(r, f).$$

Similarly,

$$T(r, G) \geq (n + m - \bar{d}(P) - Q)T(r, g) + S(r, g).$$

Also it follows from Lemmas 2.3 and 2.4 that

$$\begin{aligned} &\bar{N}(r, 0; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) \\ &\leq 2\bar{N}(r, 0; f) + 2\bar{N}(r, 0; g) + \bar{N}(r, 0; P[f]) + \bar{N}(r, 0; P[g]) \\ &\quad + \bar{N}(r, \infty; f) + \bar{N}(r, \infty; g) + S(r, f) + S(r, g) \\ &\leq \left(\frac{\bar{d}(P)(k + 1) + Q + 3}{s}\right) T(r, f) + \left(\frac{\bar{d}(P)(k + 1) + Q + 3}{s}\right) T(r, g) \\ &\quad + S(r, f) + S(r, g) \\ &\leq \left(\frac{2\bar{d}(P)(k + 1) + 2Q + 6}{(n + m - \bar{d}(P) - Q)s}\right) T(r) + S(r) \end{aligned}$$

where $S(r) = o\{T(r)\}$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure. When $s = 1, 2$, we get

$$n > m + 8\bar{d}(P) - 4\underline{d}(P) + \frac{4Q + 17}{s}$$

When $s \geq 3$, we get

$$n > m + \frac{(4k+5)\bar{d}(P) + 4Q + 17}{s}.$$

From above it follows that

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\bar{N}(r, 0; f) + \bar{N}(r, 0; g) + \bar{N}(r, \infty; f) + \bar{N}(r, \infty; g)}{T(r)} < 1$$

which gives by Lemma 2.7 either $FG \equiv 1$ or $F \equiv G$.

Subcase 1.2. Let $FG \equiv 1$.

Then by Lemma 2.8, we get a contradiction.

Subcase 1.3. $F \equiv G$, that is

$$\begin{aligned} & f^n (f^m + \dots + (-1)^i m_{C_{m-i}} f^{m-i} + \dots + (-1)^m) P[f] \\ & \equiv g^n (g^m + \dots + (-1)^i m_{C_{m-i}} g^{m-i} + \dots + (-1)^m) P[g]. \end{aligned} \quad (3.14)$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in (3.14) we get

$$\begin{aligned} & g^{n+m} (h^{n+m+\bar{d}(P)} - 1) + \dots + (-1)^i m_{C_{m-i}} g^{n+m-i} (h^{n+m-i+\bar{d}(P)} - 1) \\ & \quad + \dots + (-1)^m g^n (h^{n+\bar{d}(P)} - 1) = 0 \end{aligned}$$

which implies that $h^b = 1$, where $b = (n+m+\bar{d}(P), n+m-1+\bar{d}(P), \dots, n+\bar{d}(P))$.

Thus $f(z) = tg(z)$ for a constant t such that $t^b = 1$.

If h is not constant then we must have

$$f^n (f-1)^m P[f] \equiv g^n (g-1)^m P[g].$$

Case 2. Suppose that $l = 1$ and $H \not\equiv 0$. Then by Lemma 2.6(1), we get

$$\begin{aligned} T(r, F) & \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ & \quad + \frac{1}{2} \bar{N}(r, \infty; F) + \frac{1}{2} \bar{N}(r, 0; F) + S(r, F) + S(r, G). \end{aligned} \quad (3.15)$$

We deduce from (3.15) that

$$\begin{aligned} T(r, F) & \leq 3\bar{N}(r, 0; f) + mN(r, 0; f) + N_2(r, 0; P[f]) + 2\bar{N}(r, 0; g) + mN(r, 0; g) \\ & \quad + N_2(r, 0; P[g]) + \frac{5}{2} \bar{N}(r, \infty; f) + 2\bar{N}(r, \infty; g) \\ & \quad + \frac{1}{2} \bar{N}(r, 0; P[f]) + S(r, f). \end{aligned} \quad (3.16)$$

It follows from (3.3), (3.4) and (3.16) that,

$$\begin{aligned}
 (n + m - \bar{d}(P)) T(r, f) &\leq T(r, F) - \bar{d}(P)N(r, \infty; f) - N(r, 0; P[f]) + S(r, f) \\
 &\leq 3\bar{N}(r, 0; f) + mN(r, 0; f) + 2\bar{N}(r, 0; g) + mN(r, 0; g) \\
 &\quad + N_2(r, 0; P[g]) + \frac{5}{2}\bar{N}(r, \infty; f) + 2\bar{N}(r, \infty; g) \\
 &\quad + \frac{1}{2}\bar{N}(r, 0; P[f]) - \bar{d}(P)N(r, \infty; f) \\
 &\quad + S(r, f) + S(r, g). \tag{3.17}
 \end{aligned}$$

Using Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
 N_2(r, 0; P[g]) &\leq Q\bar{N}(r, \infty; g) + \bar{d}(P)N_{k+2}(r, 0; g) + S(r, g) \\
 &\leq \left(\frac{\bar{d}(P)(k+2) + Q}{s} \right) T(r, g) + S(r, g) \tag{3.18}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{1}{2}\bar{N}(r, 0; P[f]) &\leq \frac{1}{2} (Q\bar{N}(r, \infty; f) + \bar{d}(P)N_{k+1}(r, 0; f)) + S(r, f) \\
 &\leq \frac{1}{2} \left(\frac{\bar{d}(P)(k+1) + Q}{s} \right) T(r, f) + S(r, f). \tag{3.19}
 \end{aligned}$$

Substituting (3.18) and (3.19) into (3.17), we get

$$\begin{aligned}
 (n + m - \bar{d}(P)) T(r, f) &\leq \left(\frac{Q + \bar{d}(P)(k+1) + 11}{2s} + m - \bar{d}(P) \right) T(r, f) \\
 &\quad + \left(\frac{Q + \bar{d}(P)(k+2) + 4}{s} + m \right) T(r, g) + S(r, f). \tag{3.20}
 \end{aligned}$$

Similarly for G, we get

$$\begin{aligned}
 (n + m - \bar{d}(P)) T(r, g) &\leq \left(\frac{Q + \bar{d}(P)(k+1) + 11}{2s} + m - \bar{d}(P) \right) T(r, g) \\
 &\quad + \left(\frac{Q + \bar{d}(P)(k+2) + 4}{s} + m \right) T(r, f) + S(r, g). \tag{3.21}
 \end{aligned}$$

Combining (3.20) and (3.21), we obtain

$$\left(n - m - \frac{3Q + \bar{d}(P)(3k+5) + 19}{2s} \right) [T(r, f) + T(r, g)] \leq S(r, f) + S(r, g)$$

which contradicts the condition

$$n > m + \frac{(3k + 5)\bar{d}(P) + 3Q + 19}{2s}$$

for $s \geq 7$. Therefore $H \equiv 0$. Proceeding as in Case 1 we obtain the result of Case 2.

Case 3. Let $l \geq 2$, Using Lemma 2.5, we have

$$\begin{aligned} T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \\ + S(r, F) + S(r, G). \end{aligned} \quad (3.22)$$

We deduce from (3.22) that,

$$\begin{aligned} T(r, F) \leq 2\bar{N}(r, 0; f) + mN(r, 0; f) + N_2(r, 0; P[f]) + 2\bar{N}(r, 0; g) + mN(r, 0; g) \\ + N_2(r, 0; P[g]) + 2\bar{N}(r, \infty; f) + 2\bar{N}(r, \infty; g) \\ + S(r, f) + S(r, g). \end{aligned} \quad (3.23)$$

It follows from (3.3), (3.4) and (3.23) and Lemmas 2.3 and 2.4 that

$$\begin{aligned} (n + m - \bar{d}(P)) T(r, f) \leq T(r, F) - \bar{d}(P)N(r, \infty; f) \\ - N(r, 0; P[f]) + S(r, f) \\ \leq 2\bar{N}(r, 0; f) + mN(r, 0; f) + 2\bar{N}(r, 0; g) + mN(r, 0; g) \\ + N_2(r, 0; P[g]) + 2\bar{N}(r, \infty; f) + 2\bar{N}(r, \infty; g) \\ - \bar{d}(P)N(r, \infty; f) + S(r, f) + S(r, g). \end{aligned} \quad (3.24)$$

Now

$$\begin{aligned} N_2(r, 0; P[g]) \leq Q\bar{N}(r, \infty; g) + \bar{d}(P)N_{k+2}(r, 0; g) \\ \leq \left(\frac{\bar{d}(P)(k+2) + Q}{s} \right) T(r, g) + S(r, g). \end{aligned} \quad (3.25)$$

Substituting (3.25) into (3.24), we get

$$\begin{aligned} (n + m - \bar{d}(P)) T(r, f) \leq \left(\frac{4}{s} + m - \bar{d}(P) \right) T(r, f) \\ + \left(\frac{\bar{d}(P)(k+2) + Q + 4}{s} + m \right) T(r, g) \\ + S(r, f) + S(r, g). \end{aligned} \quad (3.26)$$

Similarly for G , we get

$$\begin{aligned} (n + m - \bar{d}(P)) T(r, g) &\leq \left(\frac{4}{s} + m - \bar{d}(P) \right) T(r, g) \\ &\quad + \left(\frac{\bar{d}(P)(k + 2) + Q + 4}{s} + m \right) T(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we get

$$\left(n - m - \frac{\bar{d}(P)(k + 2) + Q + 8}{s} \right) [T(r, f) + T(r, g)] \leq S(r, f) + S(r, g)$$

which contradicts to

$$n > m + \frac{(k + 2)\bar{d}(P) + Q + 8}{s}.$$

Hence either $FG \equiv 1$ or $F \equiv G$. So proceeding as in Case 1, we obtain the conclusion of Case 3. Hence the Theorem 1.1 is proved.

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